Optimal Control

Prob: To find the path from a to b so that J is minimum

$$J = \int_a^b F(x_1, x_2, \dots, x_n; \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t)\,dt$$

Again by Euler equation

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = 0, \quad i = 1, 2, \dots, n.$$

Where the values of x_i are given at the end points, $t = a$ & $t = b$.

Remark-1 If the end condition at $t = a$ is not given,

$$\left.\frac{\partial F}{\partial \dot{x}_i}\right|_{t=a} = 0$$, this is called transversability condition.

Optimization with Constraints;

$$J = \int_a^b F(x_1, x_2, \dots, x_n; \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t)\,dt$$

with $g(x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t) = 0$

$h(x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t) = 0$

In this case Augmented Cost functional is,

$$J^* = \int_a^b (F + \lambda g + \mu h)\,dt.$$

Prob: To find the path from a to b so that J^* is minimum.

Optimal Control problem;

$$J = \int_a^b F(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t)\,dt$$

where $u_1, u_2, \dots, u_m$ are control variables with

$$\frac{dx_i}{dt} = f_i(\underbrace{x_1, x_2, \dots, x_n}_{\text{State variable}}, \underbrace{u_1, u_2, \dots, u_m}_{\text{Control variable}}, t)$$

Ex-: Electro chemical process:

In an electro chemical process the system is modeled by the diff. Equation, $\frac{d^2x}{dt^2} = -\dot{x} + u$, where x and u are functions of t. Minimize the cost functional

$$J = \frac{1}{2}\int_0^\infty (x^2 + \alpha u^2)\,dt$$

where α is the disposal constant by choosing the control variable u appropiately.

Ans: We introduce the state variables x_1 and x_2 to reduce the given differential equation to a set of 1st order diff. equations, let $\left.x_1 = x \;\&\; x_2 = \dot{x}_1 (= \dot{x})\right\}$ —— ①

Then constraints on the state and control variables are,

$$\dot{x}_1 - x_2 = 0 \quad \text{—— ②}$$

Again from $\frac{d^2x}{dt^2} = -\dot{x} + u$ or, $\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\dot{x} + u$

We have $\frac{d}{dt}(\dot{x}_1) = -x_2 + u$

$$\text{or, } \dot{x}_2 + x_2 - u = 0 \quad \text{—— (2a)}$$

The end conditions for the electro-chemical system are that x and $\dot{x}$ are given at $t = 0$ and both tend to zero as $t \to \infty$.

Therefore, in terms of state variables, the above end conditions are $x_1(0) = a$ (say),

$$\left.\begin{array}{l} x_2(0) = \dot{x}_1(0) = b \text{ (say)}; \\ x_1(t) \to 0 \text{ and } x_2(t) \to 0 \text{ as } t \to \infty \end{array}\right\} \text{—— ③}$$

Now the given problem reduces to Min $J = \frac{1}{2}\int_0^\infty (x_1^2 + \alpha u^2)\,dt$ ~~Sut~~ Sub. to the constraints ② and (2a) with the end conditions ③.

We formulate the augmented cost functional as,

$$J^* = \int_0^\infty \left[\frac{1}{2}(x_1^2 + \alpha u^2) + \lambda(\dot{x}_1 - x_2) + \mu(\dot{x}_2 + x_2 - u) \right] dt \quad \text{——— (4)}$$

Where λ & μ are Lagrangian multipliers, x_1, x_2 are state variables and u is the control variable.

Relevant Euler's equations are:

For x_1: $\quad \dfrac{\partial F}{\partial x_1} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{x}_1}\right) = 0$

Here, $F(x_1, x_2, \dot{x}_1, \dot{x}_2, u, t) = \frac{1}{2}(x_1^2 + \alpha u^2) + \lambda(\dot{x}_1 - x_2) + \mu(\dot{x}_2 + x_2 - u)$

or, $x_1 - \dfrac{d}{dt}(\lambda) = 0 \Rightarrow x_1 = \dot{\lambda}$ ——— (5)

For x_2: $\quad \dfrac{\partial F}{\partial x_2} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{x}_2}\right) = 0$

or, $-\lambda + \mu - \dfrac{d}{dt}(\mu) = 0$ or, $\lambda = \mu - \dot{\mu}$ ——— (6)

For u: $\quad \dfrac{\partial F}{\partial u} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{u}}\right) = 0$

$\cancel{\text{or, } -\lambda + \mu - \frac{d}{dt}(\mu) = 0 \text{ or, } \lambda =}$

$\alpha u - \mu = 0 \Rightarrow \mu = \alpha u$ ——— (7)

Thus we have $\quad x_1 = \dot{\lambda},\ \lambda = \mu - \dot{\mu},\ \mu = \alpha u \}$ - -

$$\frac{d^4 x_1}{dt^4} = \frac{d^3}{dt^3}\left(\frac{dx_1}{dt}\right) = \frac{d^3 x_2}{dt^3} \quad (\text{as } x_2 = \dot{x}_1)$$

$$= \frac{d^2}{dt^2}\left(\frac{dx_2}{dt}\right) = \frac{d^2}{dt^2}(u - x_2) \quad [\text{using (2a)}]$$

$$= \ddot{u} - \frac{d^2 x_2}{dt^2} = \ddot{u} - \frac{d}{dt}\left(\frac{dx_2}{dt}\right) = \ddot{u} - \frac{d}{dt}(u - x_2)$$

$$= \ddot{u} - \dot{u} + \frac{dx_2}{dt} = \ddot{u} - \dot{u} + \frac{d}{dt}\left(\frac{dx_1}{dt}\right)$$

$$= \ddot{u} - \dot{u} + \frac{d^2 x_1}{dt^2}$$

or, $\frac{d^4 x_1}{dt^4} - \frac{d^2 x_1}{dt^2} = \frac{\ddot{u}}{\alpha} - \frac{\dot{u}}{\alpha} = \frac{1}{\alpha}\frac{d}{dt}(\dot{u} - u)$

$= \frac{1}{\alpha}\frac{d}{dt}(-\lambda)$ [by ⑥

$= -\frac{1}{\alpha}\dot{\lambda} = -\frac{1}{\alpha}x_1$ [by ⑤

or, $\frac{d^4 x_1}{dt^4} - \frac{d^2 x_1}{dt^2} + \frac{x_1}{\alpha} = 0$

or, $\left(D^4 - D^2 + \frac{1}{\alpha}\right) x_1 = 0$ —— ⑧

Clearly, the solution x_1, x_2 and the control variable u, depend on the choice of α called as " disposal constant ".

For simplicity, let $\alpha = 4$.

$\therefore$ From ⑧, $\left(D^4 - D^2 + \frac{1}{4}\right) x_1 = 0$

$\Rightarrow \left(D + \frac{1}{\sqrt{2}}\right)^2 \left(D - \frac{1}{\sqrt{2}}\right)^2 x_1 = 0$

Solving, $x_1 = (At + B)e^{-t/\sqrt{2}} + (ct + D)e^{t/\sqrt{2}}$, where A, B, c & D are constants which evaluated by the end conditions which are $x_1(0) = a$, $x_2(0) = b$,

$x_1(t) = 0$, $x_2(t) = 0$ as $t \to \infty$.

From the conditions, $x_1(t), x_2(t) \to 0$ as $t \to \infty$, we have $c = 0 = D$.

$\therefore \quad x_1(t) = (At + B)e^{-t/\sqrt{2}}$

Again $x_1(0) = a \Rightarrow B = a$.

$x_1(t) = (At + a)e^{-t/\sqrt{2}}$

$x_2(t) = \dot{x}_1(t) = Ae^{-t/\sqrt{2}} - \frac{1}{\sqrt{2}}(At + a)e^{-t/\sqrt{2}}$

$x_2(0) = b \Rightarrow A - \frac{a}{\sqrt{2}} = b \Rightarrow A = b + \frac{a}{\sqrt{2}}$

$\therefore\ x_1(t) = \left[\left(\frac{a}{\sqrt{2}}+b\right)t + a\right]e^{-t/\sqrt{2}}\ \ [\equiv x(t)]$

$x_2(t) = \frac{dx_1}{dt} = \left[b - \frac{1}{\sqrt{2}}\left(b+\frac{a}{\sqrt{2}}\right)t\right]e^{-t/\sqrt{2}}$

Now,

$u = \dot{x}_2 + x_2$ (from given differential equation)

$= -\frac{1}{\sqrt{2}}\left[-\frac{1}{\sqrt{2}}\left(\frac{a}{\sqrt{2}}+b\right)t + b\right]e^{-t/\sqrt{2}} + \left\{-\frac{1}{\sqrt{2}}\left(b+\frac{a}{\sqrt{2}}\right)e^{-t/\sqrt{2}}\right\}$

$+ \left[-\frac{1}{\sqrt{2}}\left(b+\frac{a}{\sqrt{2}}\right)t + b\right]e^{-t/\sqrt{2}}$

$= \left[\frac{1}{2}(1-\sqrt{2})\left(b+\frac{a}{\sqrt{2}}\right)t - (\sqrt{2}-1)b - \frac{a}{2}\right]e^{-t/\sqrt{2}}$

Thus we have determined the control variable $u = u(t)$ which minimizes the cost functional J. Substituting the values of x and u in J and then integrating w.r.to t, we get the minimum value of J.

Block – Diagram

The solution of these problem can be illustrated with the help of a block diagram as follows.

Let $u(t)$ to be of the form

$u(t) = a_1 x_1(t) + a_2 x_2(t)$

Where a_1, a_2 are constraints to be determined. Substituting the values of u, x_1, x_2 in

$u = a_1 x_1 + a_2 x_2$

$\left[\frac{1}{2}(1-\sqrt{2})\left(b+\frac{a}{\sqrt{2}}\right)t - (\sqrt{2}-1)b - \frac{a}{2}\right]e^{-t/\sqrt{2}} = a_1\left[\left(b+\frac{a}{\sqrt{2}}\right)t + a\right]e^{-t/\sqrt{2}} + a_2\left[-\frac{1}{\sqrt{2}}\left(b+\frac{a}{\sqrt{2}}\right)t + b\right]e^{-t/\sqrt{2}}.$

This is an identity, we get the constant terms and

~~we~~

We get $\frac{1}{\sqrt{2}}(1-\sqrt{2}) = a_1 - \frac{a_2}{\sqrt{2}}$

or, $-(\sqrt{2}-1)b - \frac{a}{2} = a a_1 + a_2 b$

Solving, $a_1 = -\frac{1}{2}$, $a_2 = -(\sqrt{2}-1)$

$\therefore$ $u(t) = a_1 x_1(t) + a_2 x_2(t)$

$= -\frac{1}{2}x_1(t) - (\sqrt{2}-1)x_2(t)$

$= -0.5x_1(t) - 0.414x_2(t)$ — (9)

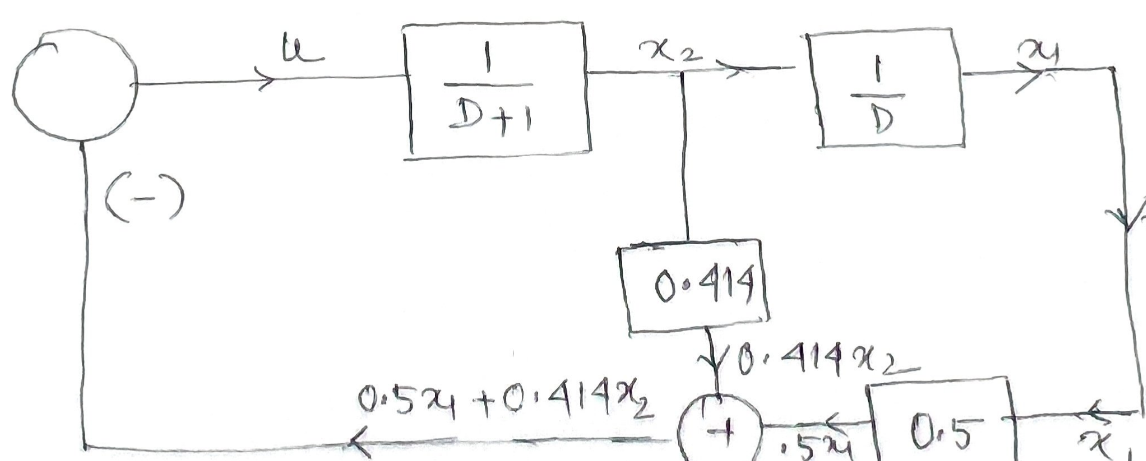
Given the values of $x_1(t)$, $x_2(t)$ at any time t, the expression (9) gives the values of $u(t)$ at that time. This may be illustrated by the following diagram.

We have, $x_1 = x$, $x_2 = \dot{x}_1 = \frac{d}{dt}(x_1) = Dx_1$,

$x_1 = -\frac{1}{D}x_2$

$\dot{x}_2 + x_2 - u = 0 \Rightarrow u = \dot{x}_2 + x_2 = (D+1)x_2$

$x_2 = \left(\frac{1}{D+1}\right)u$.



Block-diagram of the electro-chemical process.

Ex-1 An electrochemical System is characterised by $\frac{dx_1}{dt} = -x_1 + u$, $\frac{dx_2}{dt} = x_1$

where u is the control variable chosen so that as

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to minimize the cost functional

$$J = \int^{\infty} \left(x_2^2 + \frac{16}{3} u^2\right) dt.$$

Show that if the boundary condition satisfying the st[ate] variables are $x_1(0) = a$, $x_1(\infty) = 0$; $x_2(0) = b$, $x_2(\infty) =$
Then the optimum choice for u is

$u(t) = -0.366\, x_1(t) - 0.4433\, x_2(t)$.

Illustrate the feedback control in a Block Diagram.

Ans: Here $J^* = \int_0^{\infty} \left[x_2^2 + \frac{16}{3}u^2 + \lambda(\dot{x}_2 - x_1) + \mu(\dot{x}_1 + x_1 - u)\right] dt$

Eulers equations;

for x_1: $\lambda = \mu - \dot{\mu}$; for x_2: $\dot{\lambda} = 2x_2$; for u: $\mu = \frac{32}{3}$

$\therefore \dot{\lambda} = \dot{u} - \ddot{u} \Rightarrow 2x_2 = \frac{32}{3}\dot{u} - \frac{32}{3}\ddot{u}$

i.e, $\ddot{u} - \dot{u} + \frac{3}{16}x_2 = 0$

Again $u = x_1 + \dot{x}_1 = \dot{x}_2 + \ddot{x}_2$ $\Big\} \Rightarrow \dot{u} = \ddot{x}_2 + \dddot{x}_2$, $\Rightarrow \ddot{u} = \dddot{x}_2 + \ddddot{x}_2$

$\therefore \frac{d^4x_2}{dt^4} + \frac{d^3x_2}{dt^3} - \frac{d^2x_2}{dt^2} + \frac{3}{16}x_2 - \frac{d^3x_2}{dt^3} = 0$

or, $\left(D^4 - D^2 + \frac{3}{16}\right)x_2 = 0$

$\Rightarrow \left(D + \frac{\sqrt{3}}{2}\right)\left(D - \frac{\sqrt{3}}{2}\right)\left(D + \frac{1}{2}\right)\left(D - \frac{1}{2}\right)x_2 = 0$

$\Rightarrow x_2(t) = Ae^{\frac{\sqrt{3}}{2}t} + Be^{-\frac{\sqrt{3}}{2}t} + \frac{1}{2}Ce^{t/2} - \frac{1}{2}De^{-t/2}$

As, $x_1(t), x_2(t) \to 0$ as $t \to \infty \Rightarrow A = C = 0$

$x_1(t) = \dot{x}_2(t) = -\frac{\sqrt{3}}{2}Be^{-\frac{\sqrt{3}}{2}t} - \frac{D}{2}e^{-t/2}$

$x_2(t) = Be^{-\frac{\sqrt{3}}{2}t} + De^{-t/2}$

using the other boundary condition,

$x_1(0) = a$, $x_2(0) = b$.

$D = b - B$, $B = \frac{b + 2a}{1 - \sqrt{3}}$

$$D = -\frac{2a+\sqrt{3}b}{1-\sqrt{3}} \; ; \; x_1(t) = -\frac{\sqrt{3}}{2}\,\frac{b+2a}{1-\sqrt{3}}\,e^{-\frac{\sqrt{3}}{2}t} + \frac{2a+\sqrt{3}b}{1-\sqrt{3}}\cdot\frac{1}{2}\,e^{-t/2}$$

$$x_2(t) = \frac{b+2a}{1-\sqrt{3}}\,e^{-\sqrt{3}/2\,t} - \frac{2a+\sqrt{3}b}{1-\sqrt{3}}\,e^{-t/2}.$$

$$u(t) = \dot{x}_1(t) + x_1(t)$$

$$= \left(\frac{3}{4} - \frac{\sqrt{3}}{2}\right)\frac{b+2a}{1-\sqrt{3}}\,e^{-3/2\,t} + \frac{1}{4}\,\frac{2a+\sqrt{3}b}{1-\sqrt{3}}\,e^{-t/2}$$

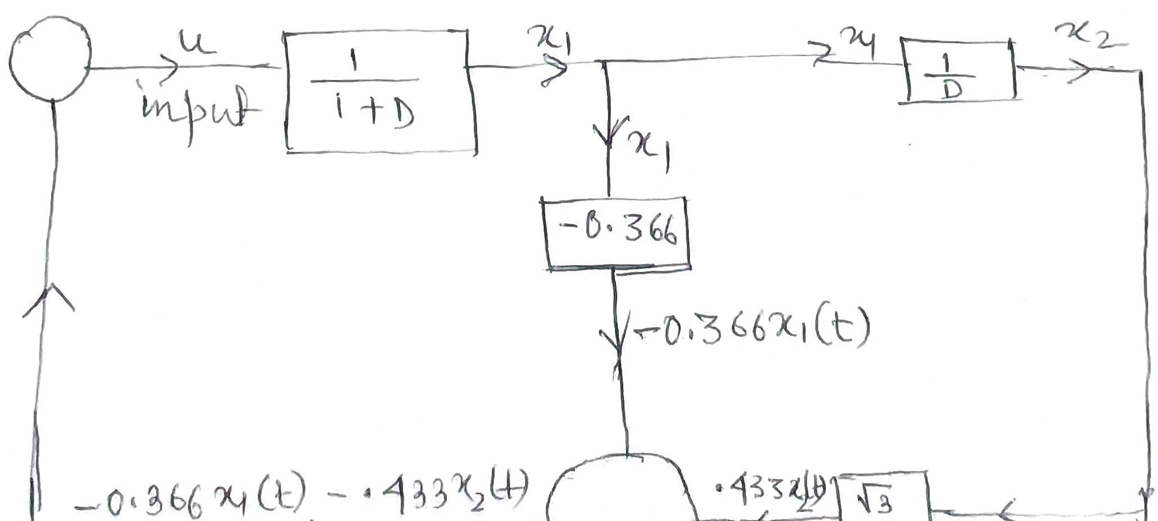
$$= a\,x_1(t) + a_2 x_2(t) \text{ (say)}$$

Putting the values of $x_1(t)$, $x_2(t)$ and then Comparing the Co-efficient of $e^{-\sqrt{3}/2\,t}$ and $e^{-t/2}$ from both sides

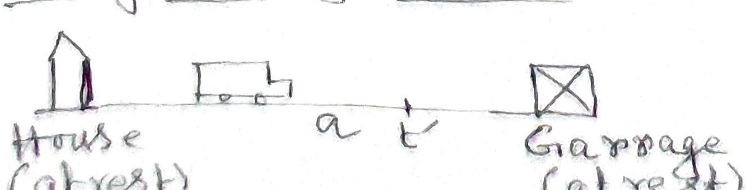
$$a_1 = \frac{1-\sqrt{3}}{2}, \quad a_2 = -\frac{\sqrt{3}}{4}.$$

$$u(t) = \frac{1-\sqrt{3}}{2}\,x_1(t) - \frac{\sqrt{3}}{4}\,x_2(t)$$

$$= -0.366\,x_1(t) - 0.433\,x_2(t)$$



Bang - Bang Control



The concept of Bang-Bang Control is illustrated by followin

Example.

Let a car is drived from a stationary position in a horizontal drive way, in a stationary position in a garrage moving a total distance, a. The available controls for the driver are (i) acceleration and (ii) Brake i.e, deaceleration, assuming there is no gear system (for simplicity).

The equation of motion is $\frac{d^2x}{dt^2} = f(t)$ —— (1)

where $f = f(t)$ represants the applied acceleration or deaceleration (i.e, brake) force.

Clearly, f will be subjected to its both lower and upper values i.e, maximum acceleration and maximum deaceleration so that $-\alpha \le f(t) \le \beta$ —— (2)

where β is the maximum acceleration and α is the maximum deaceleration possible.

This is required as the problem is to minimize the travel time period, T (say).

Now, our problem is to solve the equation (1) with constraint (2) and the end conditions.

$$x(0) = x|_{t=0} = 0 \qquad \dot{x}(0) = \dot{x}(t)|_{t=0} = 0$$
$$x(T) = x|_{t=T} = a \qquad \dot{x}(T) = \dot{x}(t)|_{t=T} = 0.$$

where T is the travel time period.

Now, the control problem is to find the value of the control variable f which accomplies the whole operation in a minimum time i.e, so that T is minimum.

Solⁿ. Time of travel, T can be expressed as

$$T = \int_0^T 1\,dt = \int_0^T \frac{dt}{dx}\,dx = \int_0^T \frac{1}{\left(\frac{dx}{dt}\right)}\,dx = \int_0^T \frac{1}{v}\,dx,$$

where $v = v(x) = \frac{dx}{dt}$ and $v(0) = 0$, $v(a) = 0$.

$$\text{Now } f = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$
$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dg}{dx}, \text{ where } g = \frac{1}{2}v^2,\ v = \sqrt{2g}.$$

$\therefore \quad \frac{dq}{dx} - f = 0$ —— ③

then, $T = \int_0^a \frac{1}{v}\, dx = \int_0^a \frac{1}{\sqrt{2g}}\, dx$ —— ④

$g(0) = \frac{1}{2} v^2(0) = 0, \quad g(a) = \frac{1}{2} v^2(a) = 0 \}$ —— ⑤

The required problem is to find the Control f which minimize the functional T given by ④ subject to the inequality constraints ② and the equality constraints ③ with the end conditions ⑤.

Transform of inequality constraints to corresponding equality constraint. Now we transform the inequality constraints ② to the equality constraint introducing a 2nd control variable z (say) where $z^2 = (f+\alpha)(-f+\beta)$ —⑥
z being a real variable.

If the inequality constraints ② is satisfied, then $z^2 \geq 0$ and z is real.

Hence, the problem is to minimize T is given by ④ sub. to the equality constraints ③ and ⑥ with the end conditions ⑤.

The augmented cost functional is

$$T^* = \int_0^a \left[\frac{1}{\sqrt{2g}} + \lambda\left(\frac{dg}{dx} - f\right) + \mu\left\{z^2 - (f+\alpha)(-f+\beta)\right\}\right] dx \quad —⑦$$

where g is the state variable, f and z are control variables, λ and μ are Lagrange Multipliers. The path will satisfy the following Euler eqn's.

for g: $\frac{\partial F}{\partial g} - \frac{d}{dx}\left(\frac{\partial F}{\partial g'}\right) = 0; \quad F = \frac{1}{\sqrt{2g}} + \lambda\left(\frac{dg}{dx} - f\right) + \mu\{z^2 - (f+\alpha)(-f+\beta)\}$

$\therefore \; -(2g)^{3/2} - \frac{d\lambda}{dx} = 0$ — ⑧

for f: $\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) = 0 \; ; \quad -\lambda + \mu(2f + \alpha - \beta) = 0$ —⑨

for z: $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0 \Rightarrow 2\mu z = 0$ — (10)

from (10), either $\mu = 0$ or $z = 0$

If $\mu = 0$, then $\lambda = 0$ [from (9)],

$g = \infty$ [from (8)] $\Rightarrow v = \infty$ as $g = \frac{1}{2}v^2$.

$T = 0$, $v = \sqrt{2g}$. which is not feasible.

$\therefore \mu \neq 0$; $\therefore z = 0 \Rightarrow (f+\alpha)(-f+\beta) = 0 \Rightarrow$ Either $f = -\alpha$ or $f = \beta$.

Thus on the optimal path, the acceleration takes either its maximum +ve value or its maximum negative value i.e, maximum value, not any intermediate value. Hence, from the nature of the problem, we can conclude that the initial acceleration force β is applied and then after same time $t = t'$ (say), the maximum deacce-leration α, i.e, force $-\alpha$ is applied to bring the car to rest (zero value) at time $t = T$.

Thus the equation of motion are

$$\frac{d^2x}{dt^2} = f = \beta, \quad 0 < t \leq t'$$
$$= -\alpha \quad t' \leq t \leq T.$$

Here, the switching to change from β to $-\alpha$ takes place at a time $t = t'$ which is to be determined.

Integrating the above equation,

$$\frac{dx}{dt} = \beta t, \text{ for } 0 \leq t \leq t'$$
$$= -\alpha(t-T) \text{ for } t' \leq t \leq T.$$

Since $\frac{dx}{dt} = 0$ at $t = 0$ & $t = T$.

Further integrating and using the end condition $x = 0$ at $t = 0$ & $x = a$ at $t = T$

$$x = \begin{cases} \frac{1}{2}\beta t^2 & \text{for } 0 \le t \le t' \\ -\frac{\alpha}{2}(t'-T)^2 + a & \text{for } t' \le t \le T \end{cases}$$

As $x(t)$ and $\frac{dx}{dt}$ are continuous at $t = t'$. We have from the above expression $\beta t'^2 = -\frac{\alpha}{2}(t'-T)^2 + a$

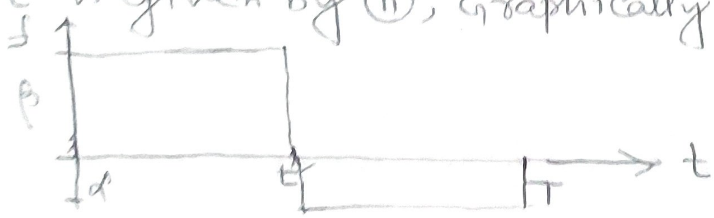
and $\beta t' = -\alpha(t'-T)$.

$$\therefore t' = \left[\frac{2a\alpha}{\beta(\alpha+\beta)}\right]^{1/2} \quad (11), \quad T = \left[\frac{2a(\alpha+\beta)}{\alpha\beta}\right]^{1/2} \quad (12)$$

Hence maximum travel time is given by $\left[\frac{2a(\alpha+\beta)}{\alpha\beta}\right]^{1/2}$ and the optimal control applied is

$$f = \beta, \text{ for } 0 \le t \le t'$$
$$= -\alpha \quad \text{for } t' \le t \le T.$$

Where t' is given by (11), Graphically, it is represented as



Ex. The angular motion of a ship is described in terms of a variable θ by the equation $\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = p$.
Where p is the rudder setting which is subjected to the constraint $|p| \le 1$. The change in course from $\theta = \alpha$, $\frac{d\theta}{dt} = 0$ to $\theta = 0$, $\frac{d\theta}{dt} = 0$ is required, find p to minimize the time for correction [Rudder ⇒ steering board of a ship]

Ans. The time taken to change the required correction is

$$T = \int_0^T dt = \int_\alpha^0 \frac{dt}{d\theta}\, d\theta = \int_\alpha^0 \frac{1}{\left(\frac{d\theta}{dt}\right)}\, d\theta = \int_\alpha^0 \frac{1}{\omega}\, d\theta$$
where $\omega = \frac{d\theta}{dt}$

$$p = \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d\theta}{dt} = \frac{d\omega}{dt} + \omega = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} + \omega$$

$$= \omega\frac{d\omega}{d\theta} + \omega = \omega\left(1+\frac{d\omega}{d\theta}\right) \quad \text{——} \quad (1)$$

Here, p is the Control variable. Now, the given problem is, to choose ω to minimize T Subject to the Constraint $p = \omega\left(1+\frac{d\omega}{d\theta}\right)$ [given by (1)] and $|p| \le 1$ or, $-1 \le p \le 1$,

This inequality constraint can be transformed to an equality constraint as $z^2 - (p+1)(-p+1) = 0$

where z is another Control variable and it is real.

The augmented Cost functional is

$$T^* = \int_{\alpha}^{0}\left[\frac{1}{\omega} + \lambda\left\{p - \omega\left(1+\frac{d\omega}{d\theta}\right)\right\} + \mu\left\{z^2 - (p+1)(-p+1)\right\}\right]d\theta.$$

where λ, μ are Lagrangian multiplier, ω is the state variable, p, z are Control variables.

The relevant Euler's equ's are

<u>For ω:</u> $\frac{\partial F}{\partial \omega} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \dot{\omega}}\right) = 0$,

$$F = \frac{1}{\omega} + \lambda\left\{p - \omega\left(1+\frac{d\omega}{d\theta}\right)\right\} + \mu\left\{z^2 - (p+1)(-p+1)\right\}$$

or, $-\frac{1}{\omega^2} - \lambda(1+\dot{\omega}) - \frac{d}{d\theta}(-\lambda\omega) = 0$

$$\lambda = -\frac{1}{\omega^2} + \dot{\lambda}\omega$$

or, $\dot{\lambda}\omega - \lambda = \frac{1}{\omega^2}$ ——— (2)

<u>For p:</u> $\frac{\partial F}{\partial p} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \dot{p}}\right) = 0$

or, $\lambda + \mu(2p) = 0 \Rightarrow \lambda = -2p\mu$ — (3)

<u>For z:</u> $\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0 \Rightarrow 2\mu z = 0$ — (4)

From (4), either $\mu = 0$ or $z = 0$

If $\mu = 0$, then $\lambda = 0$ from (3)

$\frac{1}{\omega^2} = 0$ from (2) $\Rightarrow \omega \to \infty$.

it is impossible (not feasible) as it leads to $T = 0$.

Hence $\mu \neq 0$. Then we have $z = 0$ or $(p+1)(-p+1) = 0$

$\Rightarrow$ either $p = -1$ or $p = 1$.

At $t = 0$, we have $\theta = \alpha$, $\frac{d\theta}{dt} = 0$

Initially $\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d\omega}{dt}$ must be negative from physical consideration.

Thus the optimal path must start with $p = -1$ and at the end, it will be $p = +1$.

Assuming that there is only one switch from $p = -1$ to $p = +1$. We must determine when the switch over takes place i.e, the times of change of this operation from $p = -1$ to $p = 1$.

For $p = -1$, the original differential equation is

i.e, $-1 = \omega\left(1 + \frac{d\omega}{d\theta}\right) \Rightarrow d\theta = -\frac{\omega}{1+\omega} d\omega$

Integrating, $\omega - \log(1+\omega) = -\theta + c_1$

at $\theta = \alpha$, $\omega = 0$, $\therefore c_1 = \alpha$

$\therefore \quad \theta = -\omega + \log(1+\omega) + \alpha$ —— (5)

Similarly, for $p = 1$; $1 = \omega\left(1 + \frac{d\omega}{d\theta}\right)$

$\Rightarrow d\theta = \frac{\omega}{1-\omega} d\omega = \left(-1 + \frac{1}{1-\omega}\right) d\omega$

Integrating, $\theta + c_2 = -\omega - \log(1-\omega)$

At $\theta = 0$, $\omega = 0$, $\therefore c_2 = 0$

$\therefore \theta = -\{\omega + \log(1-\omega)\}$ —— (6)

The switch over occours when the values of θ from (5) and from (6) are equal

$$-\omega + \log(1+\omega) + \alpha = -\omega - \log(1-\omega^2)$$

$$\Rightarrow \quad 1-\omega^2 = e^{-\alpha}$$

$$\Rightarrow \quad \omega = \sqrt{1-e^{-\alpha}} \quad \text{—— (7)}$$

using this values,

from (6), $\theta = -\sqrt{1-e^{-\alpha}} - \log\left(1-\sqrt{1-e^{-\alpha}}\right)$

The time required to change the operation from $p=-1$ to $p=+1$ can be determined by solving the original equation.

Ex: A control system is described by $\dot{x}+x=u$. Where $x=x(t)$ is the state variable and $u=u(t)$ is control variable. If $x=x_0$ at $t=0$ and $x=Q$ at $t=T$ determine the optimal path so that u is chos. to minimize $J=\int_0^T [(Q-x)^2+u^2]\,dt$.

Hint: The given control system is $\dot{x}+x=u$ where $x=x(t)$ is the state variable and u is the control vari-ble. The end conditions are $x=x_0$ at $t=0$

$x=Q$ at $t=T$.

The augmented cost functional is

$$J^* = \int_0^T \left[(Q-x)^2+u^2 \right] + \lambda(\dot{x}+x-u)\,dt$$

proceed